



Some identities involving the binomial sequences

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Abstract

Using the exponential generating function and the Bell polynomials, we obtain several new identities for the binomial sequences. As applications, some interesting identities are established for the Abel polynomials, exponential polynomials and factorial powers. © 2007 Elsevier B.V. All rights reserved.

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1. Introduction

A sequence $\{\varphi_n(x)\}$ of polynomials is called binomial if it satisfies the following conditions:

- (1) $\varphi_0(x) = 1$, $\varphi_1(x) = x$,
- (2) for any positive integer n , $\varphi_n(x)$ is a polynomial of degree n with $\varphi_n(0) = 0$, and
- (3) for all nonnegative integer n ,

$$\varphi_n(x+y) = \sum_{k=0}^n \binom{n}{k} \varphi_k(x) \varphi_{n-k}(y). \quad (1)$$

The Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \dots)$ in an infinite number of variables x_1, x_2, \dots , defined by (see [3])

$$\exp\left(u \sum_{m \geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n \geq 1} \frac{t^n}{n!} \left[\sum_{k=1}^n u^k B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \right],$$

or, by the series expansion:

$$\frac{1}{k!} \left(\sum_{m \geq 1} x_m \frac{t^m}{m!} \right)^k = \sum_{n \geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \dots \quad (2)$$

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Their exact expression is

$$B_{n,k}(x_1, x_2, \dots) = \sum \frac{n!}{k_1! k_2! \dots (1!)^{k_1} (2!)^{k_2} \dots} x_1^{k_1} x_2^{k_2} \dots,$$

where the summation takes place over all integers k_1, k_2, \dots , such that $k_1 + 2k_2 + \dots = n$, $k_1 + k_2 + \dots = k$, and $k_i \geq 0$, $i = 1, 2, \dots$.

In [1] Abbas and Bouroubi gave the following identity on the Bell polynomials and the binomial sequences:

$$B_{n,k}(\varphi_0(1), 2\varphi_1(1), 3\varphi_2(1), \dots) = \binom{n}{k} \varphi_{n-k}(k). \quad (3)$$

Moreover, as the applications of this identity, they deduced the following identity for the Bell numbers

$$B_{n,k}(B_0, 2B_1, 3B_2, \dots) = \binom{n}{k} \sum_{j=0}^{n-k} S(n-k, j) k^j, \quad (4)$$

where $S(n, k)$ denotes the Stirling number of the second kind, and $B_n = \sum_{j=0}^n S(n, j)$ is the Bell number. Furthermore, they recovered the known identity

$$B_{n,k}(1, 2, 3, \dots) = \binom{n}{k} k^{n-k}. \quad (5)$$

Various generalized Pascal matrices related to some certain binomial sequences were widely studied by many authors recently (see [2,5,7,8]). In [6], the properties of the lower triangular functional matrix associated with a binomial sequence are discussed, and some combinatorial identities are derived. Particularly, they proved the following identities:

$$\sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n-1}{j} \varphi_n(jx) = \frac{1}{2} n! (n-1) x^{n-2} \varphi_2(x), \quad (6)$$

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \varphi_n(jx) = \begin{cases} x^n & \text{if } n = k, \\ 0 & \text{if } 0 \leq n \leq k-1. \end{cases} \quad (7)$$

In this paper, using the exponential generating function for the binomial sequence $\{\varphi_n(x)\}$, we shall generalize the identities (3), (6) and (7) in different ways. As a consequence, we are able to find infinitely many identities by substituting special binomial sequences.

2. Some properties of the binomial sequences

Let us designate by $\Phi(x, t)$ the exponential generating function associated to the binomial sequence $\{\varphi_n(x)\}$, i.e., $\Phi(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n / n!$. Then, from identity (1), we have $\Phi(x+y, t) = \Phi(x, t) \Phi(y, t)$. Hence, for any nonnegative integer k

$$\Phi(kx, t) = (\Phi(x, t))^k. \quad (8)$$

Furthermore, the binomial sequence can be characterized via its exponential generating function as the following lemma [4, p. 87, Exercise 5.37]:

Lemma 1. Let $1 = \varphi_0(x)$, $x = \varphi_1(x)$, $\varphi_2(x), \dots$ be a sequence of polynomials, with $\deg \varphi_n(x) = n$ and $\varphi_n(0) = 0$ for all positive integers n , $\Phi(x, t) = \sum_{n=0}^{\infty} \varphi_n(x) t^n / n!$. The following three conditions are equivalent:

- (i) For all $n \in \mathbf{N}$, $\varphi_n(x+y) = \sum_{k=0}^n \binom{n}{k} \varphi_k(x) \varphi_{n-k}(y)$, where \mathbf{N} is the set of nonnegative integers.
- (ii) There exists a power series $f(t) = \sum_{n=1}^{\infty} f_n(t^n / n!)$ with $f_1 = 1$ such that

$$\Phi(x, t) = \exp(x f(t)).$$

- (iii) $\Phi(x, t) = \Phi^x(1, t)$.

Lemma 2. (i) A sequence $\{\varphi_n(x)\}$ is binomial if and only if

$$\varphi_n(x) = \sum_{k=0}^n x^k B_{n,k}(f_1, f_2, \dots, f_{n-k+1}), \quad (9)$$

where f_n is the coefficients of a power series $f(t)$ defined by $f(t) := \sum_{n=1}^{\infty} f_n(t^n/n!)$.

(ii) A sequence $\{\varphi_n(x)\}$ is binomial if and only if

$$\varphi_n(x) = \sum_{k=0}^n (x)_k B_{n,k}(\varphi_1(1), \varphi_2(1), \dots, \varphi_{n-k+1}(1)), \quad (10)$$

where $(x)_k := x(x-1)\cdots(x-k+1)$ for $k \geq 1$, and $(x)_0 := 1$.

Proof. From (9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n x^k B_{n,k}(f_1, f_2, \dots, f_{n-k+1}) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} B_{n,k}(f_1, f_2, \dots, f_{n-k+1}) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} x^k \frac{1}{k!} \left(\sum_{n=1}^{\infty} f_n \frac{t^n}{n!} \right)^k = \sum_{k=0}^{\infty} x^k \frac{1}{k!} (f(t))^k = \exp(xf(t)). \end{aligned}$$

According to Lemma 1(ii), $\{\varphi_n(x)\}$ is binomial. Conversely, if $\{\varphi_n(x)\}$ is binomial, then the generating function satisfies $\sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!} = \exp(xf(t))$, where $f(t) := \sum_{n=1}^{\infty} f_n(t^n/n!)$. With the same computation as above, we can get (9).

Analogously, from (10), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \varphi_n(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^n (x)_k B_{n,k}(\varphi_1(1), \varphi_2(1), \dots, \varphi_{n-k+1}(1)) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} (x)_k \sum_{n=k}^{\infty} B_{n,k}(\varphi_1(1), \varphi_2(1), \dots, \varphi_{n-k+1}(1)) \frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty} (x)_k \frac{1}{k!} \left(\sum_{n=1}^{\infty} \varphi_n(1) \frac{t^n}{n!} \right)^k = \left(1 + \sum_{n=1}^{\infty} \varphi_n(1) \frac{t^n}{n!} \right)^x. \end{aligned}$$

From Lemma 1(iii), $\{\varphi_n(x)\}$ is binomial. For the converse, if $\{\varphi_n(x)\}$ is binomial, then $\Phi(x, t) = \Phi^x(1, t)$, where $\Phi(x, t) := \sum_{n=0}^{\infty} \varphi_n(x) t^n/n!$. By computation, (10) can be readily derived. \square

Example 1. By Lemma 1, the following sequences are of binomial type (with $\varphi_0(x) = 1$ and with $n \geq 1$ below, c.f., [4, p. 87, 132])

- (a) $\varphi_n(x) = x^n$.
- (b) $\varphi_n(x) = (x)_n = x(x-1)\cdots(x-n+1)$.
- (c) $\varphi_n(x) = x^{n|\lambda} = x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda)$, and we may call them factorial powers.
- (d) $\varphi_n(x) = \sum_{k=0}^n s(n, k)(x)_k$, where $s(n, k)$ is the Stirling number of the first kind.
- (e) $\varphi_n(x) = \sum_{k=0}^n S(n, k)x^k$, where $S(n, k)$ is the Stirling number of the second kind. They are called exponential polynomials.
- (f) $\varphi_n(x) = x(x-na)^{n-1}$, they are the Abel polynomials.

3. Main results

Theorem 1. Let $\{\varphi_n(x)\}$ be a binomial sequence, then for all integers $n \geq k \geq 0$,

$$B_{n,k}(\varphi_0(x), 2\varphi_1(x), 3\varphi_2(x), \dots) = \binom{n}{k} \varphi_{n-k}(kx). \quad (11)$$

Proof. On the one hand,

$$\begin{aligned} \frac{1}{k!} (t\Phi(x, t))^k &= \frac{1}{k!} \left(t \sum_{n \geq 0} \varphi_n(x) \frac{t^n}{n!} \right)^k = \frac{1}{k!} \left(\sum_{n \geq 1} n \varphi_{n-1}(x) \frac{t^n}{n!} \right)^k \\ &= \sum_{n \geq k} B_{n,k}(\varphi_0(x), 2\varphi_1(x), 3\varphi_2(x), \dots) \frac{t^n}{n!}. \end{aligned}$$

On the other hand by (8),

$$\frac{1}{k!} (t\Phi(x, t))^k = \frac{1}{k!} t^k \Phi(kx, t) = \frac{1}{k!} \sum_{n \geq 0} \varphi_n(kx) \frac{t^{n+k}}{n!} = \sum_{n \geq k} \binom{n}{k} \varphi_{n-k}(kx) \frac{t^n}{n!},$$

hence the assertion follows. \square

If we choose $x = 1$ in (11) we recover identity (3) obtained in [1].

Theorem 2. Let $\{\varphi_n(x)\}$ be a binomial sequence, then for all integers $n, k \geq 0$,

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \varphi_n(jx) = B_{n,k}(\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots). \quad (12)$$

Proof. On the one hand,

$$\frac{1}{k!} (\Phi(x, t) - 1)^k = \frac{1}{k!} \left(\sum_{n \geq 1} \varphi_n(x) \frac{t^n}{n!} \right)^k = \sum_{n \geq k} B_{n,k}(\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots) \frac{t^n}{n!}.$$

On the other hand by (8), we have

$$\begin{aligned} \frac{1}{k!} (\Phi(x, t) - 1)^k &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (\Phi(x, t))^j \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \Phi(jx, t) \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{n \geq 0} \varphi_n(jx) \frac{t^n}{n!} \\ &= \sum_{n \geq 0} \left(\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \varphi_n(jx) \right) \frac{t^n}{n!}, \end{aligned}$$

and the result follows. \square

Since $B_{n,n}(\varphi_1(x), \varphi_2(x), \varphi_3(x), \dots) = (\varphi_1(x))^n = x^n$, (12) yields (7) when $0 \leq n \leq k$. If putting $k = n - 1$ in (12), we get (6), because $B_{n,n-1}(\varphi_1(x), \varphi_2(x)) = \frac{1}{2}n(n-1)\varphi_1^{n-2}(x)\varphi_2(x) = \frac{1}{2}n(n-1)x^{n-2}\varphi_2(x)$.

Applying (10) and (12), we get

$$\varphi_n(x) = \sum_{k=0}^n (x)_k \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \varphi_n(j). \quad (13)$$

Theorem 3. Let $\{\varphi_n(x)\}$ be a binomial sequence, then for all integers $n, k \geq 0$,

$$\begin{aligned} & \frac{1}{k!} \sum_{j=0}^k \sum_{i=0}^n (-1)^{k-j} \binom{k}{j} \binom{n}{i} \binom{k-j}{i} i! x^i \varphi_{n-i}(jx) \\ &= (n)_k B_{n-k,k} \left(\frac{1}{2} \varphi_2(x), \frac{1}{3} \varphi_3(x), \frac{1}{4} \varphi_4(x), \dots \right) \\ &= B_{n,k}(0, \varphi_2(x), \varphi_3(x), \varphi_4(x), \dots). \end{aligned} \quad (14)$$

Proof. On the one hand,

$$\begin{aligned} \frac{1}{k!} (\Phi(x, t) - 1 - xt)^k &= \frac{1}{k!} \left(\sum_{n \geq 2} \varphi_n(x) \frac{t^n}{n!} \right)^k = \frac{t^k}{k!} \left(\sum_{n \geq 1} \varphi_{n+1}(x) \frac{t^n}{(n+1)!} \right)^k \\ &= t^k \sum_{n \geq k} B_{n,k} \left(\frac{1}{2} \varphi_2(x), \frac{1}{3} \varphi_3(x), \frac{1}{4} \varphi_4(x), \dots \right) \frac{t^n}{n!} \\ &= \sum_{n \geq 2k} (n)_k B_{n-k,k} \left(\frac{1}{2} \varphi_2(x), \frac{1}{3} \varphi_3(x), \frac{1}{4} \varphi_4(x), \dots \right) \frac{t^n}{n!}. \end{aligned}$$

On the other hand by (8), we have

$$\begin{aligned} \frac{1}{k!} (\Phi(x, t) - 1 - xt)^k &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (\Phi(x, t))^j (1 + xt)^{k-j} \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \Phi(jx, t) (1 + xt)^{k-j} \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{m \geq 0} \varphi_m(jx) \frac{t^m}{m!} \sum_{i=0}^{k-j} \binom{k-j}{i} x^i t^i \\ &= \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{n \geq 0} \sum_{i=0}^n \binom{n}{i} \binom{k-j}{i} i! x^i \varphi_{n-i}(jx) \frac{t^n}{n!} \\ &= \sum_{n \geq 0} \left(\frac{1}{k!} \sum_{j=0}^k \sum_{i=0}^n (-1)^{k-j} \binom{k}{j} \binom{n}{i} \binom{k-j}{i} i! x^i \varphi_{n-i}(jx) \right) \frac{t^n}{n!}, \end{aligned}$$

and the first equality in (14) follows. In view of [3, p. 136, Equation (31')] we have

$$B_{n-k,k} \left(\frac{x_2}{2}, \frac{x_3}{3}, \dots \right) \frac{n!}{(n-k)!} = B_{n,k}(0, x_2, x_3, \dots).$$

Hence the second equality in (14) follows. \square

4. Applications

Example 2. Let $\varphi_n(x) = x^n$, then applying (11), (12) and (14), respectively, we have

$$B_{n,k}(1, 2x, 3x^2, 4x^3, \dots) = \binom{n}{k} (kx)^{n-k}, \quad (15)$$

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (jx)^n = B_{n,k}(x, x^2, x^3, \dots), \quad (16)$$

$$\begin{aligned} \frac{1}{k!} \sum_{j=0}^k \sum_{i=0}^n (-1)^{k-j} \binom{k}{j} \binom{n}{i} \binom{k-j}{i} i! j^{n-i} x^n \\ = (n)_k B_{n-k,k} \left(\frac{1}{2}x^2, \frac{1}{3}x^3, \frac{1}{4}x^4, \dots \right) = B_{n,k}(0, x^2, x^3, x^4, \dots). \end{aligned} \quad (17)$$

Example 3. Let

$$\varphi_n(x) = x^{n|\lambda} = \begin{cases} x(x+\lambda)(x+2\lambda)\dots(x+(n-1)\lambda) & \text{if } n \geq 1, \\ 1 & n = 0, \end{cases}$$

then the sequence is binomial, where λ is an arbitrary parameter. Applying (11), (12) and (14), respectively, we have

$$B_{n,k}(1, 2x, 3x^{2|\lambda}, 4x^{3|\lambda}, \dots) = \binom{n}{k} (kx)^{(n-k)|\lambda}, \quad (18)$$

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (jx)^{n|\lambda} = B_{n,k}(x, x^{2|\lambda}, x^{3|\lambda}, \dots), \quad (19)$$

$$\begin{aligned} \frac{1}{k!} \sum_{j=0}^k \sum_{i=0}^n (-1)^{k-j} \binom{k}{j} \binom{n}{i} \binom{k-j}{i} i! x^i (jx)^{(n-i)|\lambda} \\ = (n)_k B_{n-k,k} \left(\frac{1}{2}x^{2|\lambda}, \frac{1}{3}x^{3|\lambda}, \dots \right) = B_{n,k}(0, x^{2|\lambda}, x^{3|\lambda}, \dots). \end{aligned} \quad (20)$$

In particular for $x = \lambda$, identity (18) gives (see [1, p. 8, Remark 5])

$$B_{n,k}(1, 2\lambda, 3!\lambda^2, 4!\lambda^3, \dots) = \lambda^{n-k} \binom{n-1}{k-1} \frac{n!}{k!}. \quad (21)$$

Setting $x = \lambda$ in (19), we get

$$B_{n,k}(\lambda, 2!\lambda^2, 3!\lambda^3, \dots) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (j+n-1)_n \lambda^n. \quad (22)$$

Combining Eq. (21) with Eq. (22), we get an identity when $\lambda = 1$:

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (j+n-1)_n = \binom{n-1}{k-1} \frac{n!}{k!} = L(n, k), \quad (23)$$

where $L(n, k)$ are the Lah numbers (see, e.g., [3, p.135] and [5]).

Example 4. Let $\varphi_n(x) = x(x - na)^{n-1}$ be the Abel polynomials, then the sequence is binomial (see [3, p. 129] and [4, p. 132]). Applying (11), (12) and (14), respectively, we have

$$B_{n,k}(1, 2x, 3x(x - 2a), 4x(x - 3a)^2, \dots) = \binom{n}{k} (kx)(kx - (n - k)a)^{n-k-1}, \quad (24)$$

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} jx(jx - na)^{n-1} = B_{n,k}(x, x(x - 2a), x(x - 3a)^2, \dots), \quad (25)$$

$$\begin{aligned} \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{i=0}^n \binom{n}{i} \binom{k-j}{i} i! x^i jx(jx - (n-i)a)^{n-i-1} \\ = (n)_k B_{n-k,k} \left(\frac{1}{2}x(x - 2a), \frac{1}{3}x(x - 3a)^2, \dots \right) = B_{n,k}(0, x(x - 2a), x(x - 3a)^2, \dots). \end{aligned} \quad (26)$$

In particular for $x = -a$, identity (24) gives

$$B_{n,k}(1, -2a, (3a)^2, -(4a)^3, \dots) = \binom{n-1}{k-1} (-na)^{n-k}. \quad (27)$$

Example 5. Let $B_n(x) = \sum_{j=0}^n S(n, j)x^j$ be the ordinary generating function of the Stirling numbers of the second kind $S(n, k)$. The sequence $\{B_n(x)\}$ is called exponential polynomials (see [4,9]), and it is of binomial type. Applying (11), (12) and (14), respectively, we have

$$B_{n,k}(B_0(x), 2B_1(x), 3B_2(x), \dots) = \binom{n}{k} B_{n-k}(kx), \quad (28)$$

$$\frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} B_n(jx) = B_{n,k}(B_1(x), B_2(x), B_3(x), \dots), \quad (29)$$

$$\begin{aligned} \frac{1}{k!} \sum_{j=0}^k \sum_{i=0}^n (-1)^{k-j} \binom{k}{j} \binom{n}{i} \binom{k-j}{i} i! x^i B_{n-i}(jx) \\ = (n)_k B_{n-k,k} \left(\frac{1}{2}B_2(x), \frac{1}{3}B_3(x), \frac{1}{4}B_4(x), \dots \right) = B_{n,k}(0, B_2(x), B_3(x), \dots). \end{aligned} \quad (30)$$

With the substitution $x=1$, (28) gives (4). Since $(1/k!) \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} B_n(jx) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \sum_{i=0}^n S(n, i)x^i$
 $(jx)^i = \sum_{i=0}^n S(n, i)x^i (1/k!) \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^i = \sum_{i=0}^n S(n, i)x^i S(i, k)$, we have

$$\sum_{i=0}^n S(n, i)x^i S(i, k) = B_{n,k}(B_1(x), B_2(x), B_3(x), \dots), \quad n \geq k. \quad (31)$$

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